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Convergence Results in a Well-Known Delayed Predator-Prey System

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In this paper, we provide a detailed and explicit procedure of obtaining some regions of attraction for the positive steady state (assumed to exist) of a well known Lotka–Volterra type predator-prey system with a single discrete delay. Our procedure requires the delay length to be small. A detailed example is presented. The method used here is to construct a proper Liapunov functional in a restricted region.

1. INTRODUCTION

We consider the following well-studied Lotka–Volterra type predator-prey system with a single discrete delay [5, 8], where $x(t)$, $y(t)$ stand for the population density of prey and predator at time t, respectively,

$$\begin{aligned} x'(t) &= x(t)(r - ax(t) - by(t)), \\ y'(t) &= y(t)(-d + cx(t-\tau)). \end{aligned} \tag{1.1}$$

Here r, a, b, c, d, τ are positive constants (for relevant background of this system, see [5, p. 247]). It is well known that if (1.1) has no positive

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equilibrium, then the boundary equilibrium $E_0 = (r/a, 0)$ is globally asymptotically stable with respect to initial conditions

$$x_0(\theta) = \phi(\theta) \geq 0, \qquad \theta \in [-\tau, 0], \qquad x(0) > 0, \qquad y(0) > 0, \quad (1.2)$$

where $\phi \in C([-\tau, 0], \mathbf{R}^+)$, $\mathbf{R}^+ = \{x : x \geq 0\}$, and $\|\phi\| = \max\{|\phi(\theta)| : \theta \in [-\tau 0]\}$. If

$$d/c < r/a, \tag{1.3}$$

then (1.1) has a positive equilibrium $E^* = (x^*, y^*)$, where

$$x^* = d/c, \qquad y^* = (r - ad/c)/b. \tag{1.4}$$

It is also well known that if $\tau = 0$ and (1.3) holds, then E^* is globally asymptotically stable with respect to the positive cone $\operatorname{int} \mathbf{R}_2^+ = \{(x, y) : x > 0,\ y > 0\}$ (see [10, Chap. 2]). However, when $\tau > 0$, E^* may or may not be even locally stable. It is easy to show that (see [5, p. 74]) there is a $\tau_* = \tau_*(r, a, b, c, d) > 0$ such that if $\tau < \tau_*$, then E^* is locally stable and if $\tau > \tau_*$, then E^* is unstable. Indeed, it is also known that $\tau > \tau_*$ together with some very reasonable conditions ensures the existence of a nonconstant positive periodic solution for some much more general kind of delayed predator-prey systems ([11, 9] and also in [5, p. 247]). Nevertheless, up to this date despite the efforts of many experienced researchers, there are no results obtained (1) for the global asymptotical stability of the positive equilibrium E^* when τ is small enough, or (2) for the convergence results with respect to E^* when τ is small (no results of any kind exist).

The objective of this paper is to provide some relevant results to the above two questions.

Global stability results for delayed differential systems are numerous. However, most of them require the considered system to satisfy the so-called diagonal instantaneous negative feedback dominance conditions [5, p. 215]. Recently there are some results where such dominance requirements can be significantly weakened [7, 6]; however, these results need the existence of at least some delayed diagonal negative feedbacks. All these requirements are too luxurious for system (1.1). It is fair to say that existing techniques do not work very well here.

Our approach is the theoretically simple but practically difficult and tedious Liapunov functional method. The construction of our Liapunov functional is divided into three essential parts, each serving some important roles. Even with this much work and luck, we must admit that our results are somewhat complicated and therefore not very satisfactory. Still, we are pleased that some meaningful convergence results are now available for system (1.1).

2. PRELIMINARIES

We assume in the rest of this paper that (1.3) holds, that is, (1.1) has a positive equilibrium $E^* = (x^*, y^*)$, with $x^* = d/c$ and $y^* = (r - a\,dc^{-1})b^{-1}$. If we define

$$u_1 = (x - x^*)/x^* \qquad \text{and} \qquad u_2 = (y - y^*)/y^*, \tag{2.1}$$

or, equivalently,

$$x = x^* + x^* u_1, \qquad \text{and} \qquad y = y^* + y^* u_2, \tag{2.2}$$

then (1.1) becomes

$$\begin{aligned} u_1'(t) &= -x^{*-1}x(t)\big(Au_1(t) + Bu_2(t)\big), \\ u_2'(t) &= Cy^{*-1}y(t)u_1(t-\tau), \end{aligned} \tag{2.3}$$

where

$$A = ax^*, \qquad B = by^*, \qquad C = cx^*. \tag{2.4}$$

Note that we have purposely left x, y in (2.3). One can always substitute x, y by $x^* + x^* u_1$, $y^* + y^* u_2$, respectively, if necessary.

It is well known [5, p. 247] that solutions of (2.1) and (2.2) uniquely exist and stay positive for all $t > 0$. Also, the proof of Theorem 2.1 in [5, p. 247] shows that there is a positive constant M, such that all solutions of (1.1) and (1.2) satisfy

$$\limsup_{t \to +\infty} x(t) \le ra^{-1}, \qquad \limsup_{t \to +\infty} y(t) \le M \tag{2.5}$$

Indeed, it is easy to see that for large t, $x(t) < ra^{-1}$ when (1.3) holds.

The following two lemmas will be useful in the proof of our main results in the next section.

Consider an autonomous system of delay differential equation

$$\dot{x}(t) = F(x_t) \tag{2.6}$$

such that $F(0) = 0$ and $F: C([-\tau, 0], \mathbf{R}^n) \to \mathbf{R}^n$, $\tau > 0$, is Lipschitzian, where $C = C([-\tau, 0], \mathbf{R}^n)$ is the set of continuous functions defined on $[-\tau, 0]$ with the norm $\|\phi\| = \max_{\theta \in [-\tau, 0]} |\phi(\theta)|$, and where $|\cdot|$ is any norm in $\mathbf{R}^n$. The following lemma needs no proof.

LEMMA 2.1. *Let $\omega_1(\cdot)$ and $\omega_2(\cdot)$ be nonnegative continuous scalar functions such that $\omega_i(0) = 0$, $i = 1, 2$; $\omega_2(r) > 0$ for $r > 0$, $\lim_{r \to \infty} \omega_1(r) = +\infty$ and $V: C \to R$ is a continuously differentiable scalar functional that for a special set S of solutions of* (2.6), *the following are satisfied*

$$V(\phi) \ge \omega_1(|\phi(0)|), \qquad \dot{V}(\phi)|_{(2.6)} \le -\omega_2(|\phi(0)|). \tag{2.7}$$

Then $x = 0$ is asymptotically stable with respect to the set S. That is, solutions that stay in S converge to $x = 0$.

In the process of the construction of a key Liapunov functional in the next section, we will need the solution of the following optimization problem: Let m, n, p, q, and σ be positive constants and w, β be two positive variables satisfying

$$\beta > \sigma \qquad \text{and} \qquad w = \sigma/(\beta - \sigma). \tag{2.8}$$

Find

$$\eta = \max\left\{\min\left\{\frac{w}{p + qw\beta}, \frac{1}{m + nw\beta}\right\}\right\}. \tag{2.9}$$

LEMMA 2.2. *Let $\mu = m + (n+q)\sigma + [(m + 2p + (n+q)\sigma)^2 - 4(m + p)(p + q\sigma)]^{1/2}$, then the solution to the above optimization problem is $\eta = 2\mu^{-1}$.*

Proof. By substituting $w = \sigma/(\beta - \sigma)$ into (2.9), we arrive at

$$\eta = \max\left\{\min\left\{\frac{\beta - \sigma}{(m + n\sigma)\beta - m\sigma}, \frac{\sigma}{(p + q\sigma)\beta - p\sigma}\right\}\right\}. \tag{2.10}$$

Let

$$f(\beta) = \frac{\beta - \sigma}{(m + n\sigma)\beta - m\sigma}, \qquad g(\beta) = \frac{\sigma}{(p + q\sigma)\beta - p\sigma}. \tag{2.11}$$

It is easy to see that for $\beta \geq \sigma$, f is strictly increasing while g is strictly decreasing. $f(\sigma) = 0 < g(\sigma) = 1/(q\sigma)$. Hence the solution of (2.10) with respect to $\beta \geq \sigma$ is the unique solution of

$$f(\beta) = g(\beta), \qquad \beta > \sigma, \tag{2.12}$$

which is the larger root β_+ of

$$(p + q\sigma)\beta^2 - \left[(m + 2p)\sigma + (n + q)\sigma^2\right]\beta + (m + p)\sigma^2 = 0. \tag{2.13}$$

Let

$$\mu = m + (n+q)\sigma + \left[(m + 2p + (n+q)\sigma)^2 - 4(m + p)(p + q\sigma)\right]^{1/2},$$

then

$$\beta_+ = \frac{\sigma(\mu + 2p)}{2(p + q\sigma)}, \qquad \eta = \frac{2}{\mu}.$$

∎

3. MAIN RESULTS

In this section we shall show by constructing a proper Liapunov functional, that for small enough delay length τ, we can find an *explicitly* defined region G in the neighborhood of the positive equilibrium E^* which is a subset of the basin of attraction of E^*. We would like to stress here that one should not equate this with the usual local asymptotic stability results, where basins of attraction for locally asymptotically stable steady states are guaranteed to exist only *implicitly*.

We shall use system (2.3) in our analysis below. Note that for $t \geq 0$, $1 + u_i(t) > 0$, $i = 1, 2$.

Consider first the following scalar function $V_0(t)$, which is defined as

$$V_0(t) \equiv V_0\big(u_1(t), u_2(t)\big) \equiv \ln\big(1 + u_1(t)\big) + \alpha \ln\big(1 + u_2(t)\big). \quad (3.1)$$

Here α is a positive constant whose value is to be determined later. For convenience, we let

$$z_i \equiv z_i\big(u_i(t)\big) \equiv \ln\big(1 + u_i(t)\big), \qquad i = 1, 2. \quad (3.2)$$

We then have $V_0 = z_1 + \alpha z_2$. It is easy to see that

$$\frac{d}{dt}\left(\frac{1}{2}V_0^2\right) = V_0 V_0' = (z_1 + \alpha z_2)\big(-Au_1 - Bu_2 + \alpha C u_1(t - \tau)\big).$$

Using the fact that

$$u_1(t - \tau) = u_1(t) - \int_{t-\tau}^{t} u_1'(s)\, ds, \quad (3.3)$$

we obtain

$$\begin{aligned}\left(\tfrac{1}{2}V_0^2\right)' = {} & -Az_1u_1 - Bz_1u_2 + \alpha C u_1 z_1 - \alpha A z_2 u_1 \\ & - \alpha B z_2 u_2 + \alpha^2 C z_2 u_1 - \alpha C(z_1 + \alpha z_2)\int_{t-\tau}^{t} u_1'(s)\, ds.\end{aligned}$$

By choosing $\alpha = AC^{-1}$, we have

$$\left(\tfrac{1}{2}V_0^2\right)' = -\alpha B z_2 u_2 - B z_1 u_2 - \alpha C(z_1 + \alpha z_2)\int_{t-\tau}^{t} u_1'(s)\, ds. \quad (3.4)$$

Observe that for $u_i \neq 0$, $z_i u_i > 0$. Indeed, $z_2 u_2$ behaves somewhat like the term u_2^2. This enables the term $-\alpha B z_2 u_2$ in (3.4) to play a key role in the control of terms such as $u_1 u_2$, $z_1 u_2$, $u_1 z_2$, and $z_1 z_2$. In order to control these terms, clearly we also need terms like $-u_1^2$ or $-z_1 u_1$ or $-z_1^2$. This will be accomplished by the following scalar function V_1,

$$V_1 \equiv V_1(u_1(t), u_2(t)) \equiv u_1 - z_1 + \beta(u_2 - z_2), \tag{3.5}$$

where β is a positive constant to be determined later. We have

$$\begin{aligned} V_1' &= -Au_1^2 - Bu_1u_2 + \beta C u_2\left[u_1 - \int_{t-\tau}^{t} u_1'(s)\,ds\right] \\ &= -Au_1^2 + (\beta C - B)u_1u_2 - \beta C u_2 \int_{t-\tau}^{t} u_1'(s)\,ds. \end{aligned}$$

Hence we have

$$\begin{aligned} \left(\tfrac{1}{2}V_0^2 + wV_1\right)' = &-\alpha B z_2 u_2 - wAu_1^2 + [w(\beta C - B)u_1 - Bz_1]u_2 \\ &+ \alpha C(z_1 + \alpha z_2)\int_{t-\tau}^{t} x(s)[Au_1(s) + Bu_2(s)]\,ds \\ &+ w\beta C u_2 \int_{t-\tau}^{t} x(s)[Au_1(s) + Bu_2(s)]\,ds. \end{aligned}$$

Observe that

$$\begin{aligned} z_1\int_{t-\tau}^{t} x(s)u_1(s)\,ds &= \int_{t-\tau}^{t} z_1(t)\cdot x(s)u_1(s)\,ds \\ &\le \tfrac{1}{2}z_1^2\tau + \tfrac{1}{2}\int_{t-\tau}^{t} x^2(s)u_1^2(s)\,ds, \end{aligned}$$

and by similar manipulations for other integral terms, we obtain that

$$\begin{aligned} &\alpha C(z_1 + \alpha z_2)\int_{t-\tau}^{t} x(s)[Au_1(s) + Bu_2(s)]\,ds \\ &\quad + w\beta C u_2 \int_{t-\tau}^{t} x(s)[Au_1(s) + Bu_2(s)]\,ds \\ &\quad \le \tfrac{1}{2}\alpha C(A+B)z_1^2\tau + \tfrac{1}{2}\alpha^2 C(A+B)z_2^2\tau \\ &\qquad + \tfrac{1}{2}w\beta C(A+B)u_2^2\tau + P\int_{t-\tau}^{t} x^2(s)u_1^2(s)\,ds \\ &\qquad + Q\int_{t-\tau}^{t} x^2(s)u_2^2(s)\,ds, \end{aligned}$$

where

$$P \equiv \tfrac{1}{2}CA\left(\alpha + \alpha^2 + w\beta\right)$$
$$Q \equiv \tfrac{1}{2}CB\left(\alpha + \alpha^2 + w\beta\right) = BP/A. \tag{3.6}$$

Therefore,

$$\begin{aligned}\left(\tfrac{1}{2}V_0^2 + wV_1\right)' \leq {} & -\alpha B z_2 u_2 - wAu_1^2 + \left[w(\beta C - B)u_1 - Bz_1\right]u_2 \\ & + \tfrac{1}{2}\alpha C(A+B)z_1^2\tau + \tfrac{1}{2}\alpha^2 C(A+B)z_2^2\tau \\ & + \tfrac{1}{2}w\beta C(A+B)u_2^2\tau + P\int_{t-\tau}^{t} x^2(s)u_1^2(s)\,ds \\ & + Q\int_{t-\tau}^{t} x^2(s)u_2^2(s)\,ds. \end{aligned} \tag{3.7}$$

In order to have some negative definite expression in the right-hand side of the above inequality, we need to find ways to control the last two integral terms. We consider

$$\begin{aligned} V_2 \equiv V_2(\phi_1, \phi_2) \equiv {} & P\int_{t-\tau}^{t} ds \int_s^t \left[x^*(1+\phi_1(\nu))\phi_1(\nu)\right]^2 d\nu \\ & + Q\int_{t-\tau}^{t} ds\int_s^t \left[x^*(1+\phi_1(\nu))\phi_2(\nu)\right]^2 d\nu, \end{aligned}$$

which is equivalent to (for (2.3))

$$V_2 \equiv P\int_{t-\tau}^{t} ds \int_s^t x^2(\nu)u_1^2(\nu)\,d\nu + Q\int_{t-\tau}^{t} ds \int_s^t x^2(\nu)u_2^2(\nu)\,d\nu. \tag{3.8}$$

We have

$$\begin{aligned} V_2' = {} & Px^2(t)u_1^2(t)\tau - P\int_{t-\tau}^{t} x^2(s)u_1^2(s)\,ds \\ & + Qx^2(t)u_2^2(t)\tau - Q\int_{t-\tau}^{t} x^2(s)u_2^2(s)\,ds. \end{aligned} \tag{3.9}$$

We can now define our Liapunov functional V on $\{(\phi_1, \phi_2) : \phi_1 \in C([-\tau, 0], \mathbf{R}),\ \phi_2(\theta) \equiv \phi(0),\ \theta \in [-\tau, 0]\}$ as

$$\begin{aligned} V &\equiv V(\phi_1, \phi_2) \\ &\equiv \tfrac{1}{2}V_0\left(\phi_1(0), \phi_2(0)\right)^2 + wV_1\left(\phi_1(0), \phi_2(0)\right) + V_2(\phi_1, \phi_2). \end{aligned} \tag{3.10}$$

Then, by combining (3.7) and (3.9), we obtain

$$\begin{aligned} V' \le\ & -\alpha B z_2 u_2 - wAu_1^2 + \left[w(\beta C - B)u_1 - Bz_1 \right] u_2 \\ & + \tfrac{1}{2}\alpha C(A+B) z_1^2 \tau + \tfrac{1}{2}\alpha^2 C(A+B) z_2^2 \tau + \tfrac{1}{2} w\beta C(A+B) u_2^2 \tau \\ & + Px^2(t) u_1^2(t) \tau + Qx^2(t) u_2^2(t) \tau. \end{aligned} \tag{3.11}$$

Recall that $z_i = \ln(1+u_i)$ for $i = 1, 2$. We define for $i = 1, 2$,

$$\varepsilon_i u_i \equiv \varepsilon_i(u_i) u_i \equiv z_i - u_i. \tag{3.12}$$

By substituting $z_i = u_i + \varepsilon_i u_i$ and $x = x^*(1 + u_1)$ into (3.11), we obtain

$$\begin{aligned} V' \le\ & -\alpha B u_2^2 - \alpha B \varepsilon_2 u_2^2 - wAu_1^2 + \left[w(\beta C - B)u_1 - Bu_1 \right] u_2 \\ & - B\varepsilon_1 u_1 u_2 + \tfrac{1}{2}\alpha C(A+B) u_1^2 \left(1 + 2\varepsilon_1 + \varepsilon_1^2\right) \tau \\ & + \tfrac{1}{2}\alpha^2 C(A+B) u_2^2 \left(1 + 2\varepsilon_2 + \varepsilon_2^2\right) \tau + \tfrac{1}{2} w\beta C(A+B) u_2^2 \tau \\ & + Px^{*2}(1+u_1)^2 u_1^2 \tau + Qx^{*2}(1+u_1)^2 u_2^2 \tau \\ =\ & -\left\{ wA - \tfrac{1}{2}\left[\alpha C(A+B) + 2Px^{*2} \right] \tau \right\} u_1^2 \\ & - \left\{ \alpha B - \tfrac{1}{2}\left[C(A+B)\left(\alpha^2 + w\beta\right) + 2Qx^{*2} \right] \tau \right\} u_2^2 \\ & - \alpha B \varepsilon_2 u_2^2 + \left[w(\beta C - B) - B \right] u_1 u_2 - B\varepsilon_1 u_1 u_2 \\ & + \tfrac{1}{2}\alpha C(A+B)\left(2\varepsilon_1 + \varepsilon_1^2\right) \tau u_1^2 + \tfrac{1}{2}\alpha^2 C(A+B)\left(2\varepsilon_2 + \varepsilon_2^2\right) \tau u_2^2 \\ & + Px^{*2}\left(2u_1 + u_1^2\right) \tau u_1^2 + Qx^{*2}\left(2u_1 + u_1^2\right) \tau u_2^2. \end{aligned} \tag{3.13}$$

Notice that the first two terms and the fourth term have nothing to do with ε_1 and ε_2 and when u_1, u_2 are very small, ε_1 and ε_2 are also very small. So, in order to have the above expression negative definite for at least small values of u_1 and u_2, it is highly desirable that the coefficients of the first two terms be as negative as possible.

Recall that $\alpha = AC^{-1}$ and β and w are yet to be determined. An obvious choice is to eliminate the fourth term by suitable values of β and w. This can be done easily by using values β, w such that $\beta > BC^{-1}$ and $w = BC^{-1}/(\beta - BC^{-1})$. In order to have negative coefficients for the first two terms, we must have τ smaller than a threshold value $\bar{\tau}$ which is defined as

$$\bar{\tau} = \min\left\{ \frac{2\alpha B}{C(A+B)\left(\alpha^2 + w\beta\right) + 2Qx^{*2}}, \frac{2wA}{\alpha C(A+B) + 2Px^{*2}} \right\}. \tag{3.14}$$

Substituting the expressions P and Q in (3.6) into (3.14), we have

$$\bar{\tau} = \min\left\{ \frac{2\alpha B}{C\left[(A+B)\alpha^2 + x^{*2}B(\alpha+\alpha^2)\right] + C(Bx^{*2}+A+B)w\beta}, \right.$$
$$\left. \frac{2Aw}{\alpha C\left[(A+B) + x^{*2}A(\alpha+1)\right] + x^{*2}CAw\beta} \right\}.$$

Clearly, the ideal choice of β and w is to maximize the value of $\bar{\tau}$, provided that $\beta > BC^{-1}$ and $w = BC^{-1}/(\beta - BC^{-1})$. Hence Lemma 2.2 can be applied here and an explicit and unique choice of β^*, w^* can be made to realize the maximum value of $\bar{\tau}$, *which we denote by* τ^* *in the following*. In the rest of this paper, we assume $\beta = \beta^*$ and $w = w^*$ in the expression of V in (3.10).

By taking advantage of the value τ^*, we have from (3.13),

$$\begin{aligned} V' \le{}& -w^*A\left\{1 - \frac{\tau}{\tau^*}\right\}u_1^2 - \alpha B\left\{1 - \frac{\tau}{\tau^*}\right\}u_2^2 \\ &- \alpha B\varepsilon_2 u_2^2 - B\varepsilon_1 u_1 u_2 + \frac{1}{2}\alpha C(A+B)\left(2\varepsilon_1 + \varepsilon_1^2\right)\tau u_1^2 \\ &+ \frac{1}{2}\alpha^2 C(A+B)\left(2\varepsilon_2 + \varepsilon_2^2\right)\tau u_2^2 + Px^{*2}\left(2u_1 + u_1^2\right)\tau u_1^2 \\ &+ Qx^{*2}\left(2u_1 + u_1^2\right)\tau u_2^2. \end{aligned} \tag{3.15}$$

We are now ready to state and prove our main result in a general form. As usual, $\|u_0\| \equiv \max\{|u_1(\theta)|,\ \theta \in [-\tau, 0], |u_2(0)|\}$.

THEOREM 3.1. *For system* (1.1), *assume that* $\tau < \tau^*$; *then there is an explicitly expressible positive constant* δ *such that if* $\|u_0\| < \delta$, *then the solution of* $(u_1(t), u_2(t))$ *of* (2.3) *tends to* $(0,0)$. *Equivalently, the solution* $(x(t), y(t))$ *of the original system* (1.1) *tends to* (x^*, y^*).

Proof. Using the inequality $2u_1u_2 \le u_1^2 + u_2^2$ and (3.15), we obtain

$$\begin{aligned} V' \le{}& \left[-w^*A\left(1 - \frac{\tau}{\tau^*}\right) - \frac{B}{2}\varepsilon_1 + \frac{\alpha}{2}C(A+B)\left(2\varepsilon_1 + \varepsilon_1^2\right)\tau \right. \\ &\qquad \left. + Px^{*2}\left(2u_1 + u_1^2\right)\tau\right]u_1^2 \\ &+ \left[-\alpha B\left(1 - \frac{\tau}{\tau^*}\right) - \alpha B\varepsilon_2 - \frac{B}{2}\varepsilon_1 \right. \\ &\qquad \left. + \frac{\alpha^2}{2}C(A+B)\left(2\varepsilon_2 + \varepsilon_2^2\right)\tau + Qx^{*2}\left(2u_1 + u_1^2\right)\tau\right]u_2^2. \end{aligned} \tag{3.16}$$

It is easy to show that if $|u| < 1/4$, then

$$|\ln(1+u) - u| \le \tfrac{8}{9}|u|^2 \le |u|^2.$$

We assume below that $|u_i| < 1/4$. Hence we have $|\varepsilon_i| \le |u_i|$, $\varepsilon_i^2 \le \frac{1}{4}|u_i| \le |u_i|$, and $u_1^2 \le |u_1|$. Let $\|u\| = \max\{|u_1|, |u_2|\}$. Then we obtain

$$V' \le -\Delta_1 u_1^2 - \Delta_2 u_2^2, \tag{3.17}$$

where

$$\Delta_1 = w^* A\left(1 - \frac{\tau}{\tau^*}\right) - \left[\frac{B}{2} + \frac{3}{2}\alpha C(A+B)\tau + 3Px^{*2}\tau\right]\|u\|, \tag{3.18}$$

$$\Delta_2 = \alpha B\left(1 - \frac{\tau}{\tau^*}\right) - \left[\alpha B + \frac{B}{2} + \frac{3}{2}\alpha^2 C(A+B)\tau + 3Qx^{*2}\tau\right]\|u\|. \tag{3.19}$$

Let

$$\delta_0 = \min\left\{\frac{2w^* A(1 - \tau/\tau^*)}{B + 3\alpha C(A+B)\tau + 6Px^{*2}\tau}, \frac{2\alpha B(1 - \tau/\tau^*)}{(2\alpha+1)B + 3\alpha^2 C(A+B)\tau + 6Qx^{*2}\tau}\right\}. \tag{3.20}$$

Then we see that $\|u(t)\| < \delta_0$ for $t \ge 0$ implies that $\Delta_1 > 0$ and $\Delta_2 > 0$ and therefore $-\Delta_1 u_1^2 - \Delta_2 u_2^2$ is negative definite. Lemma 2.1 will then ensure that $\lim_{t\to\infty} u_i(t) = 0$, $i = 1, 2$, and hence $\lim_{t\to\infty} x(t) = x^*$ and $\lim_{t\to+\infty} y(t) = y^*$. Therefore, to complete the proof, we need to find δ such that if $\|u(0)\| < \delta$ then this implies that $\|u(t)\| < \delta_0$ for all $t \ge 0$. To this end, we define

$$L = \min\left\{\tfrac{1}{2}V_0^2 + w^* V_1 : \|u\| = \delta_0\right\}, \tag{3.21}$$

and the set

$$S = \big\{\phi = (\phi_1, \phi_2) : \phi_1 \in C([-\tau, 0], \mathbf{R}),\ \phi_2(\theta) \equiv \phi_2(0),$$
$$\theta \in [-\tau, 0], \max\{\|\phi_1\|, |\phi_2(0)|\} < \delta_0, \text{ and } V(\phi_1, \phi_2) < L\big\},$$

where $\|\phi_1\| = \max_{\theta\in[-\tau,0]}|\phi_1(\theta)|$. We claim that for initial data chosen from S, we must have $\|u(t)\| < \delta_0$ for all $t \ge 0$. Otherwise, there is a $t_0 > 0$, such that $\|u(t_0)\| = \delta_0$ and $\|u(t)\| < \delta_0$ for $t \in [0, t_0)$. Obviously,

$$V(u_{t_0}) \ge \tfrac{1}{2}V_0^2 + w^* V_1 \ge L. \tag{3.22}$$

However, for $t \in [0, t_0)$, we have $\|u(t)\| < \delta_0$, and hence $V'(u_t) \leq 0$, which implies that for $t \in [0, t_0)$ we must have

$$V(u_t) \leq V(u_0) < L.$$

By continuity of V, we must have

$$V(u_{t_0}) \leq V(u_0) < L,$$

a contradiction to (3.22). This proves our claim. Since V is continuous, clearly there is a $0 < \delta < \delta_0$ such that

$$S_\delta = \{(\phi_1, \phi_2) : \phi_1 \in C([-\tau, 0], R),\ \phi_2(\theta) \equiv \phi_2(0),$$
$$\theta \in [-\tau, 0], \max\{\|\phi_1\|, |\phi_2(0)|\} < \delta\} \subset S.$$

This is a desired value for δ in our theorem, and hence the end of the proof. ∎

The following result is essentially a corollary of Theorem 3.1 with an explicit expression of δ.

THEOREM 3.2. *Assume that* $\tau < \tau^*$ *in system* (1.1). *Let* δ_0 *be defined as in* (3.20),

$$\Delta = \tfrac{8}{25} w^* \min\{1, \beta^*\} / \Big\{ \tfrac{25}{32}\big[(1+\alpha)^2 + x^{*2}(P+Q)\tau^2\big] + w^*(1+\beta^*)\Big\}, \quad (3.23)$$

and

$$\delta = \min\{\Delta^{1/2}\delta_0, \tfrac{1}{4}\}.$$

Then $\|x_0 - x^*\| < x^*\delta$ *and* $|y(0) - y^*| < y^*\delta$ *implies that*

$$\lim_{t \to +\infty} (x(t), y(t)) = (x^*, y^*).$$

Proof. Since $\delta \leq 1/4$, we have for $\|\phi\| \leq 1/4$,

$$|\ln(1 + \phi_i(0))| \leq |\phi_i(0)| + |\phi_i(0)|^2 \leq \tfrac{5}{4}|\phi_i(0)|, \qquad i = 1, 2, \quad (3.24)$$

and

$$\tfrac{1}{2}\left(\tfrac{4}{5}\right)^2 |\phi_i(0)|^2 \leq \phi_i(0) - \ln(1 + \phi_i(0)) \leq |\phi_i(0)|^2, \qquad i = 1, 2. \quad (3.25)$$

Also $|x(0)| = x^*|1 + \phi_1(0)| < x^* 5/4$. Hence, we have (using the fact that $\int_{t-\tau}^t ds \int_s^t d\nu = \frac{1}{2}\tau^2$),

$$V(\phi_1, \phi_2) \leq \tfrac{1}{2}\left[\tfrac{5}{4}(1+\alpha)\|\phi\|\right]^2 + w^*(1+\beta^*)\|\phi\|^2$$
$$+ \tfrac{1}{2}\left(\tfrac{5}{4}\right)^2 (P+Q)\tau^2 \|\phi\|^2$$
$$= \Big\{ \tfrac{25}{32}\big[(1+\alpha)^2 + x^{*2}(P+Q)\tau^2\big] + w^*(1+\beta^*)\Big\}\|\phi\|^2,$$

and the value L defined by (3.21) satisfies

$$L > \omega^* \min\{1, \beta^*\} \tfrac{1}{2}\left(\tfrac{4}{5}\right)^2 \delta_0^2 = \tfrac{8}{25} w^* \min\{1, \beta^*\} \delta_0^2.$$

Hence, if

$$\|\phi\|^2 < \tfrac{8}{25} w^* \min\{1, \beta^*\} \delta_0^2 \Big/ \Big\{ \tfrac{25}{32}\left[(1+\alpha)^2 + (P+Q)\tau^2\right] + w^*(1+\beta^*) \Big\} = \Delta \delta_0^2,$$

then $\phi \in S$. Clearly, if we define $\delta = \min\{\Delta^{1/2}\delta_0, \frac{1}{4}\}$, then $S_\delta \subset S$, where S_δ is the same as that defined in the proof of Theorem 3.1. The proof is completed by noting that $\|u_0\| < \delta$ is equivalent to $\|x_0 - x^*\| < x^*\delta$ and $|y(0) - y^*| < y^*\delta$. ∎

We present below a simple example to illustrate the procedures of applying our results and to gain a better understanding of the magnitude of δ.

EXAMPLE. Consider a special case of (1.1),

$$\begin{aligned} x' &= x(t)(2 - x(t) - y(t)), \\ y' &= y(t)(-1 + x(t)(t-\tau)), \end{aligned} \tag{3.26}$$

where $a = b = c = x^* = y^* = 1$. After making the change of variable $x = 1 + u_1$, $y = 1 + u_2$, we have a special case of (2.3) with $A = B = C = 1$. Hence $\alpha = AC^{-1} = 1$ and $P = Q = 1 + \frac{1}{2}w^*\beta^*$, and

$$\bar{\tau} = \min\left\{ \frac{w}{2 + (1/2)w\beta}, \frac{1}{2 + (3/2)w\beta} \right\}, \qquad \beta > 1, w = (\beta - 1)^{-1}.$$

Following the proof of Lemma 2.2, we see that

$$f(\beta) = (\beta - 1)\big/\left(\tfrac{7}{2}\beta - 2\right), \qquad g(\beta) = \left(\tfrac{5}{2}\beta - 2\right)^{-1}.$$

Setting $f(\beta) = g(\beta)$, we obtain $\beta^* = \frac{8}{5} + \frac{2}{5}\sqrt{6}$ and $w^* = \frac{1}{3}(2\sqrt{6} - 3)$, and $\tau^* = \frac{1}{2}(\sqrt{6} - 2)(> 0.2)$. This in turn leads to $P = Q = 1 + \frac{1}{3}\sqrt{6}$. Hence the δ_0 defined by (3.20) is

$$\delta_0 = \min\left\{ \frac{2(2\sqrt{6} - 3)(1 - \tau/\tau^*)}{3 + (36 + 6\sqrt{6})\tau}, \frac{2(1 - \tau/\tau^*)}{3 + (12 + 2\sqrt{6})\tau} \right\}$$

if τ is very small, then the second number is smaller and hence will be taken as the value of δ_0. After some simple computation, we have the value of Δ defined by (3.23) as

$$\Delta = \frac{8}{75}(2\sqrt{6} - 3) \Bigg/ \left[\frac{17}{8} + \frac{4\sqrt{6}}{3} + \frac{25}{16}\left(1 + \frac{\sqrt{6}}{3}\right)\tau^2 \right].$$

Therefore the value of δ can now be determined as $\min\{\Delta^{1/2}\delta_0, 1/4\}$. To be more specific, we can, as an example, assume that $\tau = \tau^*/2$, in which case we have $\delta_0 = \sqrt{6}/12$, and $\Delta = 256(2\sqrt{6} - 3)/[25(279 + 103\sqrt{6})] \approx 0.037$. This will give δ an approximate value of 0.039.

4. DISCUSSION

The following local stability result for (1.1) can be easily derived from the material in [5, Sect. 3.3, pp. 74–77].

THEOREM 4.1. *For system* (1.1), *assume that* $dc^{-1} < ra^{-1}$. *Let* $x^* = dc^{-1}$, $y^* = (r - a\,dc^{-1})b^{-1}$, $A = ax^*$, $B = by^*$, *and* $C = cx^*$. *Let* ω *be a positive constant satisfying* $\omega^2 = (-A^2 + \sqrt{A^4 + 4B^2C^2})/2$ *and* $\theta \in [0, 2\pi)$ *such that* $\cos\theta = -\omega^2/(BC)$, $\sin\theta = A\omega/(BC)$. *Define also* $\tau_* = \theta/\omega$. *Then for* $\tau < \tau_*$, $E^* = (x^*, y^*)$ *is locally asymptotically stable and for* $\tau > \tau_*$, E^* *is unstable*.

For the example system (3.26), the value of τ_* is around 2 while the value τ^* is around 0.2. This is about 10 times in difference. Note that δ is also small compared to the value of x^*, y^* (both are 1 in the example). This, of course, shows that our bounds for τ^* and δ have a lot of room to improve.

For a general delayed Lotka–Volterra type system of the form

$$x_i' = x_i\left(r_i - a_i x_i - b_i \int_{t-\tau}^{t} x_i(s)\, d\eta(s) + \text{other terms}\right), \qquad i = 1, \ldots, n, \tag{4.1}$$

where other terms consist of linear terms of other variables, with or without delays, we call $-a_i x_i$ an instantaneous negative feedback term, provided that $a_i > 0$. If such terms are present, then often it is feasible to apply either the Liapunov functional method or Razumikhin type of argument to obtain sufficient conditions for the convergence or global stability of some steady states (see [5, Chap. 6]. If such terms are absent, then such tasks become much more difficult. In such cases the existence of delayed negative feedback terms like $b_i \int_{t-\tau}^{t} x_i(x)\, d\eta(x)$ can sometimes be helpful, provided that $b_i > 0$ and $\eta(s)$ is nondecreasing (see [6]). Similar statements are true for other well-known systems, such as some delayed chemostat equations and epidemic models (see [1–4] and the references cited therein). For the system (1.1), we do not have any of these two terms in the y equations. This may well be the primary reason that the global stability of E^* in this system is so troublesome. Problems of this kind are plenty and open for many well-studied delay systems. We would like to stress here that the solutions of such problems are important both in theory and in applications. After all, we study delayed systems because we acknowledge their universal existence and importance, which ultimately may prohibit us from assuming the existence of any instantaneous negative feedback terms and sometimes even the delayed negative feedback terms in these more realistic models.

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